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The effect of the induced mean flow on solitary waves in deep water

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Two branches of gravity–capillary solitary water waves are known to bifurcate from a train of infinitesimal periodic waves at the minimum value of the phase speed. In general, these solitary waves feature oscillatory tails with exponentially decaying amplitude and, in the small-amplitude limit, they may be interpreted as envelope-soliton solutions of the nonlinear Schrödinger (NLS) equation such that the envelope travels at the same speed as the carrier oscillations. On water of infinite depth, however, based on the fourth-order envelope equation derived by Hogan (1985), it is shown that the profile of these gravity–capillary solitary waves actually decays algebraically (like $1/x^2$) at infinity owing to the induced mean flow that is not accounted for in the NLS equation. The algebraic decay of the solitary-wave tails in deep water is confirmed by numerical computations based on the full water-wave equations. Moreover, the same behaviour is found at the tails of solitary-wave solutions of the model equation proposed by Benjamin (1992) for interfacial waves in a two-fluid system.

1. Introduction

Since Longuet-Higgins (1989) first provided numerical evidence that solitary waves are possible on water of infinite depth when surface tension is present, significant progress has been made in our understanding of gravity–capillary solitary waves. Apart from the familiar shallow-water solitary waves described by the Korteweg–de Vries (KdV) equation, there is a new class of gravity–capillary solitary waves stemming from the fact that, in the presence of surface tension, the linear-water-wave phase speed can attain a minimum at a finite wavenumber. At this wavenumber, the phase speed is equal to the group velocity and it is possible to construct small-amplitude solitary waves in the form of locally confined wavepackets whose envelope travels at the same speed as the carrier oscillations. Accordingly, these waves may also be interpreted as particular envelope-soliton solutions of the nonlinear Schrödinger (NLS) equation (Akylas 1993; Longuet-Higgins 1993).

Consistent with this interpretation, furthermore, it is expected that two branches of symmetric solitary waves bifurcate from infinitesimal periodic waves at the minimum phase speed: 'elevation' or 'depression' waves, depending on whether the peak of the envelope coincides, respectively, with a crest or a trough of the carrier oscillations.

Vanden-Broeck & Dias (1992), in fact, were able to compute solitary waves of both these types in water of infinite depth, extending the earlier numerical work of Longuet-Higgins (1989) that had focused on the depression branch. Additional numerical results were provided in finite depth by Dias, Menasce & Vanden-Broeck (1996). Moreover, Dias & Iooss (1993) constructed small-amplitude expansions of elevation and depression solitary waves with exponentially decaying oscillatory tails in water of finite depth, consistent with the predictions of the NLS equation (Akylas 1993; Longuet-Higgins 1993).

Supporting the asymptotic and numerical studies cited above, Iooss & Kirchgässner (1990) provided a rigorous existence proof, based on centre-manifold reduction, for small-amplitude symmetric solitary waves near the minimum phase speed in water of finite depth. The proof could not be extended to the infinite-depth case, however. Later, Iooss & Kirrmann (1996) managed to handle this difficulty by following a different reduction procedure which also brought out the fact that the solitary-wave tails behave differently in water of infinite depth, their decay being slower than exponential, although the precise decay rate could not be determined. By assuming the presence of an algebraic decay, Sun (1997) was able to show that the profiles of interfacial solitary waves in deep fluids must decay like $1/x^2$ at the tails. We also remark that earlier Longuet-Higgins (1989) had inferred such a decay on physical grounds for deep-water solitary waves.

In the present paper, we derive an asymptotic expression for small-amplitude gravity–capillary solitary waves in deep water that exhibits the algebraic decay (like $1/x^2$) of the tails. For this purpose, improving on the NLS equation, we use the fourth-order envelope equation derived by Hogan (1985) that accounts for the mean flow induced by a modulated wavepacket. Even though it appears as a higher-order effect in the perturbation analysis, the mean-flow contribution decays like $1/x^2$ at infinity in deep water, ultimately dominating the exponential decay of the leading-order wave profile implied by the NLS equation. This non-uniform behaviour at the tails of solitary waves in deep water is also supported by numerical computations based on the full water-wave equations.

The algebraic decay of their tails is expected to be a common feature of solitary waves in deep fluids. In the case of interfacial waves, for example, Benjamin (1992) proposed an integral–differential equation for waves in a two-fluid system, taking the lower fluid to be of infinite depth. In the presence of capillarity at the interface, the phase speed of this model evolution equation attains an extremum at a finite wavenumber where, as expected, elevation and depression solitary waves bifurcate. Following a similar asymptotic approach as in the water-wave problem, it is confirmed that the tails of these waves indeed decay like $1/x^2$ owing to the induced mean flow.

The following discussion focuses on the two symmetric elevation and depression solitary-wave branches that bifurcate from infinitesimal periodic waves at the minimum phase speed. In addition, however, as demonstrated in a recent study of the fifth-order KdV equation by Yang & Akylas (1997), there is an infinity of symmetric and asymmetric solitary-wave solution branches that bifurcate at small, but finite, amplitude near the minimum phase speed. These solitary waves comprise more than one wavepacket and can be constructed asymptotically by piecing together NLS envelope solitons. While such multi-packet solitary waves are also expected to have algebraically decaying tails in deep water, this aspect will not be pursued here.

2. Fourth-order envelope equation

Consider gravity–capillary waves on the surface of deep water $-\infty < x < \infty$, $-\infty < z \leqslant 0$. In analysing solitary-wave disturbances, we shall follow Vanden-Broeck & Dias (1992) and use dimensionless variables based on $\sigma/(\rho c^2)$ as the characteristic lengthscale and $\sigma/(\rho c^3)$ as the characteristic timescale, c being the solitary-wave speed, σ the coefficient of surface tension and ρ the fluid density; thus, the solitary-wave speed is normalized to 1. The gravity–capillary linear dispersion relation then takes the form

$$\omega^2 = |k|(\alpha + k^2), \tag{2.1}$$

where ω and k denote the frequency and wavenumber, respectively, of infinitesimal periodic waves and

$$\alpha = \frac{g\sigma}{\rho c^4} \tag{2.2}$$

is a dimensionless parameter, g being the gravitational acceleration.

As noted earlier, our interest centres on solitary waves that bifurcate from infinitesimal periodic waves when the phase and group speeds are equal, $c_p = c_g = 1$; from (2.1) it follows that this occurs at $\omega = \omega_0 = \frac{1}{2}$, $k = k_0 = \frac{1}{2}$, for $\alpha = \alpha_0 = \frac{1}{4}$. In the small-amplitude limit, these solitary waves take the form of slowly modulated wavepackets such that the envelope travels at the same speed as the wave crests (Akylas 1993; Longuet-Higgins 1993).

Accounting for nonlinear and dispersive effects correct to third order in the wave steepness, the envelope of a weakly nonlinear gravity–capillary wavepacket in deep water is governed by the NLS equation (see, for example, Djordjevic & Redekopp 1977). A more accurate envelope equation, that includes effects up to fourth order in the wave steepness, was derived by Dysthe (1979) for pure gravity wavepackets in deep water. Later, Hogan (1985), starting from Zakharov's (1968) integral equation, extended Dysthe's equation to deep-water gravity–capillary wavepackets. Apart from the leading-order nonlinear and dispersive terms present in the NLS equation, the fourth-order equation of Hogan (1985) includes certain nonlinear modulation terms as well as a non-local term that describes the coupling of the envelope with the induced mean flow. In addition to playing a significant part in the stability of a uniform wavetrain (Dysthe 1979; Hogan 1985), this mean flow turns out to be important at the tails of gravity–capillary solitary waves in deep water (see §4).

In terms of the dimensionless variables used here, the evolution equation derived by Hogan (1985) for the envelope $A(X, T)$ of the free-surface elevation,

$$\eta(x,t) = \tfrac{1}{2}\,\epsilon\,\{A\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}\} + O(\epsilon^2), \tag{2.3}$$

takes the form

$$\mathrm{i}A_T + pA_{XX} + qA^2A^* + \mathrm{i}\epsilon\,(rA_{XXX} + uA^2A^*_X + vAA^*A_X) - \epsilon kA\overline{\phi}_X\big|_{Z=0} = 0. \tag{2.4}$$

Here $X = \epsilon(x - c_g t)$, $Z = \epsilon z$, $T = \epsilon^2 t$ are scaled variables that describe the wavepacket modulations in a frame of reference moving with the group velocity c_g. As expected, to leading order in the wave steepness $\epsilon \ll 1$, equation (2.4) reduces to the familiar NLS equation, while the coupling with the induced mean flow mentioned earlier is reflected in the last term of (2.4). Specifically, the mean-flow velocity potential $\epsilon^2\,\overline{\phi}(X, Z, T)$ satisfies the boundary-value problem

$$\begin{aligned} \overline{\phi}_{XX} + \overline{\phi}_{ZZ} &= 0 \qquad (-\infty < Z < 0,\ -\infty < X < \infty),\\ \overline{\phi}_Z &= \tfrac{1}{2}\,\omega(|A|^2)_X \quad (Z = 0),\\ |\nabla\overline{\phi}| &\to 0 \qquad (Z \to -\infty), \end{aligned}$$

from which it follows that

$$\overline{\phi}_X\big|_{Z=0} = -\tfrac{1}{2}\,\omega \int_{-\infty}^{\infty} |s|\, \mathrm{e}^{\mathrm{i}sX}\, \mathscr{F}(|A|^2)\, \mathrm{d}s, \tag{2.5}$$

where

$$\mathscr{F}(\cdot) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sX}\, (\cdot)\, \mathrm{d}X$$

denotes the Fourier transform. Hence, the coupling of the envelope with the induced mean flow enters via a non-local term in the fourth-order envelope equation (2.4). The coefficients of the rest of the terms in (2.4) are given by the following expressions:

$$p = \frac{\omega}{8k^2}\, \frac{3k^4 + 6\alpha k^2 - \alpha^2}{(\alpha + k^2)^2}, \tag{2.6a}$$

$$q = -\frac{\omega k^2}{16}\, \frac{2k^4 + \alpha k^2 + 8\alpha^2}{(\alpha - 2k^2)(\alpha + k^2)}, \tag{2.6b}$$

$$r = -\frac{\omega}{16k^3}\, \frac{(\alpha - k^2)(k^4 + 6\alpha k^2 + \alpha^2)}{(\alpha + k^2)^3}, \tag{2.6c}$$

$$u = \frac{\omega k}{32}\, \frac{(\alpha - k^2)(2k^4 + \alpha k^2 + 8\alpha^2)}{(\alpha - 2k^2)(\alpha + k^2)^2}, \tag{2.6d}$$

$$v = -\frac{3\omega k}{16}\, \frac{4k^8 + 4\alpha k^6 - 9\alpha^2 k^4 + \alpha^3 k^2 - 8\alpha^4}{(\alpha - 2k^2)^2\, (\alpha + k^2)^2}, \tag{2.6e}$$

where ω satisfies the dispersion relation (2.1).

3. Solitary waves

Envelope-soliton solutions of equation (2.4) are now sought in the form

$$A = R(X) \exp\{\mathrm{i}(\lambda T + \epsilon f(X))\} \tag{3.1}$$

with $R \to 0$ as $X \to \pm\infty$.

Upon substitution of (3.1) into (2.4), it is found that R and f satisfy the equation system

$$pR_{XX} - \lambda R + qR^3 - \epsilon k R \overline{\phi}_X\big|_{Z=0} = 0, \tag{3.2a}$$

$$pRf_{XX} + 2pR_X f_X + rR_{XXX} + (u + v)\, R^2 R_X = 0. \tag{3.2b}$$

To leading order in ϵ, the relevant solution of equation (3.2*a*) is

$$R = \operatorname{sech}\left\{ \left(\frac{q}{2p}\right)^{1/2} X \right\} \tag{3.3}$$

with $\lambda = \frac{1}{2}\, q$, so envelope solitons are possible only when $qp > 0$. Assuming this to be the case, f is then determined from equation (3.2*b*) after multiplying with R and integrating once:

$$f = -\frac{rq}{4p^2}X - \left(\frac{2p}{q}\right)^{1/2}\frac{p(u+v)-3rq}{4p^2}\tanh\left\{\left(\frac{q}{2p}\right)^{1/2}X\right\}. \tag{3.4}$$

Combining (3.1), (3.2) and (3.4) with (2.3), the envelope-soliton solution found above describes a locally confined wavepacket with envelope of permanent form. As a whole, however, this disturbance is not a solitary wave because the envelope travels with the group velocity

$$c_g = \frac{\omega}{2k}\frac{\alpha+3k^2}{\alpha+k^2}, \tag{3.5}$$

while the carrier oscillations travel with the (linear) phase speed $c_p = \omega/k$, slightly modified by nonlinear effects,

$$\frac{\omega}{k} - \frac{q}{2k}\epsilon^2.$$

A solitary wave is obtained only when these two speeds are equal, which requires that

$$c_g(k;\alpha) = 1, \tag{3.6a}$$

$$\omega(k;\alpha) - \tfrac{1}{2}q\,\epsilon^2 = k, \tag{3.6b}$$

in view of the fact that the solitary-wave speed has been normalized to 1.

As already noted, for $\epsilon = 0$, conditions (3.6) are met at $k = k_0 = \frac{1}{2}$, $\omega = \omega_0 = \frac{1}{2}$ for $\alpha = \alpha_0 = \frac{1}{4}$, where $c_p = c_g = 1$. Moreover, from (2.6), the corresponding values of the coefficients of the envelope equation (2.4) are

$$p_0 = \tfrac{1}{2}, \qquad q_0 = \frac{11}{(16)^2}, \qquad r_0 = u_0 = 0, \qquad v_0 = \tfrac{3}{32}. \tag{3.7}$$

Note that $p_0q_0 > 0$ so the envelope-soliton solution (3.3) holds, and one expects branches of solitary waves to bifurcate from infinitesimal periodic waves at k_0, ω_0 and α_0.

To describe the bifurcating solitary-wave solution branches for $\epsilon \ll 1$, we expand

$$k = k_0 + \epsilon^2 k_1 + \cdots, \qquad \alpha = \alpha_0 + \epsilon^2\alpha_1 + \cdots; \tag{3.8}$$

upon substitution into conditions (3.6), (3.6a) requires, correct to $O(\epsilon^2)$,

$$\left.\frac{\partial c_g}{\partial \alpha}\right|_0 \alpha_1 + \left.\frac{\partial^2\omega}{\partial k^2}\right|_0 k_1 = 0.$$

From (3.5), however, $(\partial c_g/\partial\alpha)|_0 = 0$ so $k_1 = 0$, and (3.6b) yields

$$\left.\frac{\partial\omega}{\partial\alpha}\right|_0 \alpha_1 = \frac{q_0}{2};$$

hence, on using (2.1) and (3.7),

$$\alpha_1 = \frac{11}{(16)^2}. \tag{3.9}$$

Returning to (3.3) and (3.4), taking into account (3.7), the magnitude of the envelope of the solitary wave and the correction to the phase read

$$R = \operatorname{sech}\left(\frac{\surd 11}{16}X\right), \tag{3.10a}$$

$$f = -\frac{3}{4\surd 11}\ \tanh\left(\frac{\surd 11}{16}\,X\right). \tag{3.10b}$$

Inserting these expressions into (3.1) and (2.3), the solitary-wave profile, correct to $O(\epsilon)$, is expressed as

$$\eta = \frac{16}{\surd 11}\,\mu^{1/2}\,\mathrm{sech}\left\{\mu^{1/2}\,(x-t)\right\}\cos\tfrac{1}{2}\,(x-t) + O(\mu), \tag{3.11}$$

where, in view of (3.9),

$$\mu = \frac{11}{(16)^2}\,\epsilon^2$$

denotes the small departure of the parameter α from its value $\alpha_0 = \frac{1}{4}$ at the bifurcation point. According to the definition of α in (2.2), the fact that $\mu > 0$ implies that the bifurcating solitary waves travel at a speed less than the minimum gravity–capillary phase speed.

We remark in passing that expression (3.11) corresponds to a symmetric elevation solitary wave, where the peak of the envelope coincides with a crest of the carrier oscillations. Based on the fourth-order envelope equation used above, it would appear that one could add an arbitrary phase shift to the carrier oscillations relative to the envelope, suggesting that asymmetric waves are also possible as this phase-shift parameter is varied. However, a more careful perturbation analysis, that accounts for exponentially small corrections in ϵ, reveals that only symmetric solitary waves – either elevation or depression for which the phase shift is equal to 0 or π, respectively – bifurcate at infinitesimal amplitude (Yang & Akylas 1997).

4. Behaviour at the solitary-wave tails

According to the leading-order expression (3.11), the solitary-wave profile features oscillatory tails with exponentially decaying amplitude. The $O(\mu)$ corrections, not explicitly displayed in (3.11), include a second-harmonic contribution proportional to $\mathrm{sech}^2\,\{\mu^{1/2}\,(x-t)\}\cos(x-t)$ and a correction to the first harmonic, proportional to $\sinh\{\mu^{1/2}\,(x-t)\}\,\mathrm{sech}^2\,\{\mu^{1/2}\,(x-t)\}\sin\frac{1}{2}\,(x-t)$, that derives from the phase f in (3.10b). Both these terms decay exponentially at the solitary-wave tails.

In addition, however, there is a contribution from the induced mean flow. More specifically, according to (2.5) and (3.1), the mean-flow velocity potential is given by

$$\epsilon^2\,\overline{\phi}\big|_{Z=0} = \mathrm{i}\,\frac{64}{11}\,\mu\int_{-\infty}^{\infty}\mathrm{sgn}\,s\,\exp\left\{\mathrm{i}s\,\frac{16}{\surd 11}\,\mu^{1/2}\,(x-t)\right\}\mathscr{F}(R^2)\,\mathrm{d}s\,,$$

and the associated free-surface elevation is

$$\overline{\eta} = -\frac{\epsilon^2}{\alpha_0}\,\overline{\phi}_t\big|_{Z=0} = -\left(\frac{16}{\surd 11}\,\mu^{1/2}\right)^3\int_{-\infty}^{\infty}|s|\exp\left\{\mathrm{i}s\,\frac{16}{\surd 11}\,\mu^{1/2}\,(x-t)\right\}\mathscr{F}(R^2)\,\mathrm{d}s.$$

Using Watson's lemma (Bender & Orszag 1978, p. 263), the asymptotic behaviour of $\overline{\eta}$ at the solitary-wave tails is found to be

$$\overline{\eta} \sim \frac{32}{\surd 11}\,\mu^{1/2}\,\mathscr{F}(R^2)\Big|_{s=0}\,\frac{1}{(x-t)^2} \qquad (\mu^{1/2}\,|x-t| \gg 1),$$

where, from (3.3),

$$\mathscr{F}(R^2)\big|_{s=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{sech}^2\left\{\frac{\surd 11}{16}\,X\right\}\mathrm{d}X = \frac{16}{\pi\surd 11}.$$

Therefore, finally,

$$\overline{\eta} \sim \frac{512}{11\pi}\,\mu^{1/2}\,\frac{1}{(x-t)^2} \qquad (\mu^{1/2}\,|x-t| \gg 1), \tag{4.1}$$

indicating that the contribution of the mean flow decays algebraically in the far field.

Comparing the exponential decay of the leading-order profile (3.11) with the algebraic decay of the mean flow (4.1) at the solitary-wave tails, it is clear that the latter ultimately dominates when $|x-t|$ is large enough:

$$|x-t| \gg -\frac{\ln\mu}{\mu^{1/2}}.$$

This non-uniform behaviour occurs in deep water only and it is caused by the non-local coupling of the envelope with the induced mean flow noted earlier. Similar non-uniformities are to be expected in general for solitary waves with oscillatory tails in deep fluids, and an example of interfacial waves in a two-fluid system will be discussed in §6 based on a model equation proposed by Benjamin (1992).

5. Numerical results

In this section, we present results from numerical solutions of the full water-wave problem, in an effort to confirm the predictions of the asymptotic theory regarding the behaviour of the tails of solitary waves in deep water.

The asymptotic results are expected to be valid for small values of μ, close to the bifurcation point, and in making a quantitative comparison with numerical computations of solitary waves, one faces a trade off: as $\mu \to 0$, the oscillations at the tails decay very slowly, masking the underlying algebraic decay of the induced mean flow unless a suitably large computational domain is used. For moderate values of μ, on the other hand, the algebraic decay of the solitary-wave tails is evident, but the asymptotic results – the constant multiplying the $1/(x-t)^2$ behaviour in (4.1) – are not in quantitative agreement with the computations.

The numerical results presented here were obtained following the numerical procedure of Vanden-Broeck & Dias (1992). Specifically, the free-surface elevation η was computed as a function of the velocity potential ϕ at the free surface by discretizing, and then solving via Newton iteration, a system of two nonlinear integral–differential equations at N equally spaced grid points. To avoid spurious results, as emphasized in Dias *et al.* (1996), the number of points N and the grid spacing $\Delta\phi$ have to be chosen with care. For our purposes, after some experimentation, it was concluded that, for $\Delta\phi = 0.09$, $N = 1800$ were enough points to give accurate solutions for values of $\mu \geqslant 0.003$. In accordance with the remarks made above, reducing μ any further would require increasing the computational domain (larger value of N).

Figure 1 shows a plot of $\ln|\eta|$ against $\ln|x-t|$ for the depression solitary wave corresponding to $\mu = 0.02$ ($\alpha = 0.27$). Note that, in this log–log plot, the free-surface elevation approaches a straight line with slope -2 to a very good approximation, supporting the conclusion that the tail ultimately decays algebraically like $1/(x-t)^2$.

To estimate the constant C multiplying $1/(x-t)^2$, we make use of the property

$$\int_{-\infty}^{\infty} \eta(\xi)\,\mathrm{d}\xi = 0$$

that, as shown by Longuet-Higgins (1989), is obeyed by gravity–capillary solitary

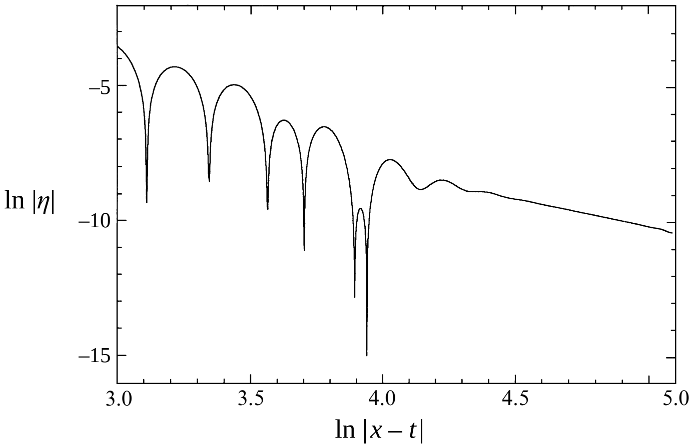

FIGURE 1. Log–log plot of the free-surface elevation $|\eta|$ as a function of $|x-t|$ for the depression solitary-wave profile corresponding to $\alpha = 0.27$.

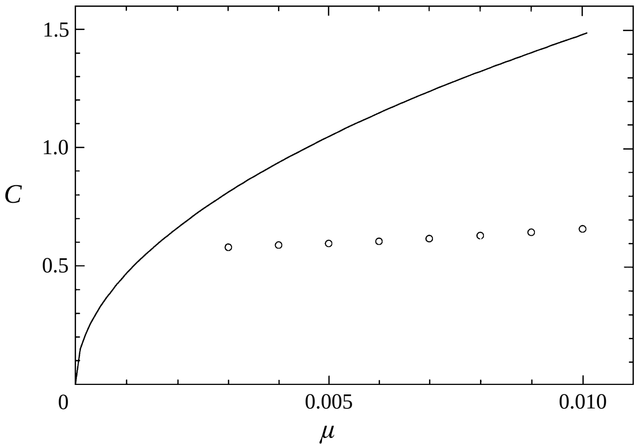

FIGURE 2. Plot of the constant C multiplying the $1/(x-t)^2$ behaviour at the tails of depression solitary waves in deep water, as a function of the parameter μ. ○: computed values on the basis of the full water-wave equations; ——: asymptotic expression according to (4.1).

waves in deep water. Assuming that the algebraic decay takes over beyond some point $x - t = \xi_\infty$, say, close to the edge of the computational domain, one then has

$$\frac{C}{\xi_\infty} = -\int_0^{\xi_\infty} \eta(\xi)\,\mathrm{d}\xi.$$

Computed values of C for depression solitary waves corresponding to certain $\mu \geqslant 0.003$ are plotted in figure 2, where the asymptotic result (4.1)

$$C \sim \frac{512}{11\pi}\,\mu^{1/2}$$

is also shown for comparison. As expected, the agreement between the asymptotic and numerical results improves as μ is decreased, but, for reasons already explained, it is difficult to extend the computations to even smaller values of μ.

6. Interfacial solitary waves

Benjamin (1992) derived an approximate evolution equation for weakly nonlinear long-wave disturbances at the interface of a two-fluid system assuming that the lower fluid is deep and the interface is subject to capillarity. This model equation consists of the KdV equation modified with the addition of a non-local dispersive term analogous to that of the Benjamin–Davis–Ono (BDO) equation. In dimensionless variables, the equation proposed by Benjamin (1992) may be written in the form

$$u_t + uu_x + a\mathscr{L}u_x + bu_{xxx} = 0. \tag{6.1}$$

The non-local operator $\mathscr{L}$ is defined as

$$\mathscr{L}(\cdot) = \int_{-\infty}^{\infty} |s|\, \mathrm{e}^{\mathrm{i}sx}\, \mathscr{F}(\cdot)\, \mathrm{d}s$$

in terms of the Fourier transform $\mathscr{F}(\cdot)$, and the parameters $a, b > 0$ are such that both dispersive terms are equally important.

In more recent work, Benjamin (1996) demonstrated that the evolution equation (6.1) admits periodic- and solitary-wave solutions. He also noted that the tails of solitary waves decay algebraically for all $a > 0$, drawing attention to the fact that, in the limit that the non-local dispersive term is relatively small, solitary waves of equation (6.1) approach those of the KdV equation, which have exponentially decaying tails, in a non-uniform manner.

In discussing solitary waves, it is convenient to scale u with c, x with $a/(2c)$ and t with $a/(2c^2)$, c being the solitary-wave speed. In terms of these scaled variables, equation (6.1) then becomes

$$u_t + uu_x + 2\mathscr{L}u_x + \alpha u_{xxx} = 0, \tag{6.2}$$

where

$$\alpha = \frac{4bc}{a^2},$$

and the solitary-wave speed is equal to 1, consistent with the normalization used earlier.

The corresponding linear dispersion relation is

$$\omega = 2k\,|k| - \alpha k^3, \tag{6.3}$$

from which it is easy to check that, for $\alpha = \alpha_0 = 1$, the phase speed $c_p = \omega/k$ attains the maximum value $c_p = 1$ at $k = k_0 = 1$. In fact, consistent with our earlier findings for gravity–capillary solitary waves, the solitary waves of equation (6.2) discussed by Benjamin (1996) bifurcate from infinitesimal periodic waves at this wavenumber and, in the small-amplitude limit, they may be interpreted as envelope solitons with stationary crests.

While the presence of an extremum of the phase speed at a finite wavenumber is essential for the bifurcation of these solitary waves, the induced mean flow, which determines the behaviour of their tails, depends on the low-wavenumber limit of the dispersion relation. More specifically, like the dispersion relation (2.1) of deep-water gravity–capillary waves, relation (6.3) is singular at $k = 0$ owing to the assumption that the lower fluid is deep, and the induced mean flow is expected to decay algebraically at the solitary-wave tails. On the other hand, when the dispersion relation has no singularity at $k = 0$, as is the case for gravity–capillary waves on water of finite depth and in the fifth-order KdV equation (Grimshaw, Malomed & Benilov 1994), the induced mean flow decays exponentially and so does the solitary-wave profile.

Following an asymptotic procedure similar to that used earlier for deep-water gravity–capillary solitary waves, we now examine small-amplitude solitary waves of the Benjamin equation (6.2) and verify that the induced mean flow indeed causes their tails to decay algebraically (like $1/x^2$).

We begin by deriving an evolution equation, analogous to (2.4), for the envelope $A(X,T)$ of a small-amplitude wavepacket. To this end, we expand

$$u = \tfrac{1}{2}\,\epsilon\,\{A(X,T)\,e^{i\theta} + \text{c.c.}\} + \epsilon^2\,\{A_2(X,T)\,e^{2i\theta} + \text{c.c.}\} + \epsilon^2\,A_0(X,T) + \cdots, \tag{6.4}$$

where $\theta = kx - \omega t$ and $X = \epsilon(x - c_g t)$, $T = \epsilon^2 t$ denote the envelope variables, c_g being the group velocity, as before.

Substituting expansion (6.4) into equation (6.2) and collecting terms proportional to $\exp(2i\theta)$, the amplitude of the second harmonic is given by

$$A_2 = \frac{1}{8}\,\frac{k}{\omega - 4k^2 + 4\alpha k^3}\,A^2 - \frac{i\epsilon}{16}\,\frac{\omega - 5\alpha k^3}{(\omega - 4k^2 + 4\alpha k^3)^2}\,(A^2)_X + O(\epsilon^2). \tag{6.5}$$

Similarly, collecting mean terms yields

$$A_0 = \frac{1}{4c_g}\,|A|^2 + \frac{\epsilon}{4c_g^2}\int_{-\infty}^{X} (|A|^2)_T\,dX' + \frac{\epsilon}{2c_g^2}\,\mathscr{L}_0(|A|^2) + O(\epsilon^2), \tag{6.6}$$

where

$$\mathscr{L}_0(\cdot) = \int_{-\infty}^{\infty} |s|\,e^{isX}\,\mathscr{F}(\cdot)\,ds.$$

Finally, collecting terms proportional to $\exp(i\theta)$ and making use of (6.5) and (6.6), it is found that $A(X,T)$ satisfies an evolution equation analogous to (2.4):

$$iA_T + pA_{XX} + qA^2A^* + i\epsilon(rA_{XXX} + uA^2A^*_X + v|A|^2A_X) - \epsilon\,\frac{k}{2c_g^2}\,A\,\mathscr{L}_0(|A|^2) = 0, \tag{6.7}$$

where

$$p = 2 - 3\alpha k, \tag{6.8a}$$

$$q = -\frac{k}{8c_g}\,\frac{2\omega - 4k^2 + 5\alpha k^3}{\omega - 4k^2 + 4\alpha k^3}, \tag{6.8b}$$

$$r = \alpha, \tag{6.8c}$$

$$u = \frac{kp}{4c_g^2} + \frac{q}{k}, \tag{6.8d}$$

$$v = \frac{k}{8}\,\frac{(\omega - 5\alpha k^3)}{(\omega - 4k^2 + 4\alpha k^3)^2} - \frac{kp}{4c_g^2} + \frac{2q}{k}. \tag{6.8e}$$

It is clear that equation (6.7) admits envelope-soliton solutions of the form (3.1), (3.3) and (3.4). These packets correspond to solitary waves of the Benjamin equation (6.2), however, only if conditions (3.6), which ensure that the envelope travels at the same speed as the wave crests, are met. As expected, for $\epsilon = 0$, these conditions are satisfied at the bifurcation point $\omega_0 = k_0 = 1$ for $\alpha_0 = 1$. Proceeding then as before, to examine solitary-wave solutions near this point, we expand k and α as in (3.8) and, upon substitution of these expansions into (3.6), making use of the dispersion relation (6.3), we find

$$k_1 = -\tfrac{9}{32}\,, \qquad \alpha_1 = \tfrac{3}{16}. \tag{6.9}$$

From (6.8), the values of the coefficients of the envelope equation (6.4) at the bifurcation point are

$$p_0 = -1, \qquad q_0 = -\tfrac{3}{8}, \qquad r_0 = 1, \qquad u_0 = \tfrac{1}{8}, \qquad v_0 = \tfrac{1}{2}.$$

Therefore, according to (3.3) and (3.4),

$$R = \operatorname{sech}\left(\frac{\surd 3}{4}\,X\right),$$

$$f = \frac{3}{32}\,X - \frac{1}{2\surd 3}\,\tanh\left(\frac{\surd 3}{4}\,X\right).$$

Combining these expressions with (3.1) and (6.4), the solitary-wave profile, correct to $O(\epsilon)$, then reads

$$u = \frac{4}{\surd 3}\,\mu^{1/2}\operatorname{sech}\{\mu^{1/2}(x-t)\}\cos(x-t) + O(\mu), \tag{6.10}$$

where, in view of (6.9), $\mu \equiv \alpha - \alpha_0 = 3\epsilon^2/16$.

Based on the leading-order expression (6.10), the solitary-wave tails feature exponentially decaying oscillations, and it is easy to check that the second harmonic in (6.5) also decays exponentially. On the other hand, using (6.7), the mean contribution (6.6) may be rewritten as

$$A_0 = \tfrac{1}{4}\,|A|^2 - \tfrac{1}{4}\mathrm{i}\epsilon\,(A^* A_X - A A_X^*) + \tfrac{1}{2}\epsilon\mathscr{L}_0(|A|^2).$$

Clearly, the first two terms above decay exponentially but the third one decays algebraically since

$$\mathscr{L}_0(R^2) = \int_{-\infty}^{\infty} |s|\,\mathrm{e}^{\mathrm{i}sX}\mathscr{F}(R^2)\,\mathrm{d}s \sim -\,2\mathscr{F}(R^2)\big|_{s=0}\,\frac{1}{X^2} \qquad (X \gg 1)$$

with

$$\mathscr{F}(R^2)\big|_{s=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{sech}^2\left(\frac{\surd 3}{4}\,X\right)\mathrm{d}X = \frac{4}{\pi\surd 3}.$$

Therefore, finally,

$$\overline{u} = \epsilon^2 A_0 \sim -\frac{16}{3\pi}\,\mu^{1/2}\,\frac{1}{(x-t)^2} \qquad (\mu^{1/2}\,|x-t| \gg 1). \tag{6.11}$$

Expression (6.11) is analogous to (4.1) derived earlier for gravity–capillary solitary waves in deep water. In both cases, the mean contribution ultimately dominates at the solitary-wave tails which, as a result, decay algebraically rather than exponentially. Similar phenomena are to be expected in general for solitary waves in deep fluids.

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